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SELF-SIMILAR SOLUTIONS OF NAVIER—STOKES EQUATIONS FOR ROTATIONAL FLOW OF INCOMPRESSIBLE FLUID IN A ROUND PIPE*

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Self-similar solutions of Navier—Stokes equations are obtained for rotational and laminar flows in a round pipe with porous and impermeable walls. Longitudinal and circular velocity components are assumed to be linear functions of the longitudinal coordinate. The obtained system of ordinary differential equations is numerically integrated. Solutions for laminar flows are not unique in the investigated here range of parameter variation that defines the intensity of suction and injection of fluid. Solutions for rotational and laminar flows in pipes with porous walls, which are characteristic of the boundary value problem, are treated separately. Curves of radial distribution of velocity components of rotational and laminar flows are presented for several intensities of sucking and injection. Curves and tables of integral characteristics of flow are also presented.

Self-similar solutions of the Navier—Stokes equations for flows in a plane-parallel channel and of laminar flows in round pipes appeared in /1/ and /2/, respectively, on the assumption of porous channel and pipe walls and of the stream function being linearly dependent on the longitudinal coordinate. The self-similar solutions derived below are for rotational flows in round pipes with porous and impermeable walls. New solutions are obtained for laminar flows in pipes with impermeable walls.

1. Assuming the flow in a round pipe of radius a to be steady and axisymmetric, we write the Navier—Stokes equations in the cylindrical system of coordinates x, r, ε (with the x axis coinciding with the pipe axis) in the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}-\frac{w^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{vw}{r}=\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{w}{r^2}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}=0$$

where u, v, w are projections of the velocity vector on axes x, r, ε.

We impose on solutions of system (1.1) the following boundary conditions along the r coordinate:

$$\partial u/\partial r=v=w=0,\quad r=0 \tag{1.2}$$

$$u=w=0,\quad v=-v_0=\text{const},\quad r=a \tag{1.3}$$

where v_0 is the velocity of injection through the pipe walls.

No conditions are imposed on the x coordinate, since our aim is to obtain a self-similar solution.

2. We seek a solution of the problem of the form

$$u=\frac{\nu x}{a^2}f(\eta),\quad w=\frac{\nu x}{a^2}g(\eta),\quad \eta=\frac{r}{a} \tag{2.1}$$

We introduce the stream function ψ and write down

$$\psi=\int_0^r ru\,dr=\nu x\varphi(\eta),\quad \varphi(\eta)=\int_0^\eta \eta f\,d\eta \tag{2.2}$$

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$$v = -\frac{1}{r}\frac{\partial\psi}{\partial x} = -\frac{\nu}{a}\frac{\varphi}{\eta} \qquad (2.3)$$

Solutions of the form (2.1) were considered in /2/ in the case of laminar flow $g(\eta) = 0$. Using (2.1) — (2.3) from Eq.(1.1) we obtain

$$-\frac{a^4}{\rho\nu^2 x}\frac{\partial p}{\partial x} = S(\eta)$$

from which

$$p = -\frac{\rho\nu^2}{a^2}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 S(\eta) + T(\eta)\right] \qquad (2.4)$$

Substituting (2.4) into the second of Eqs.(1.1) we obtain formulas that relate S and T to functions f and g

$$S' = -\frac{2g^2}{\eta}, \quad T' = \frac{1}{2}\left[\left(\frac{\varphi}{\eta}\right)^2\right]' + \frac{1}{\eta^2}\left\{\eta^2\left[\left(\frac{\varphi}{\eta}\right)' - \frac{\varphi}{\eta^2}\right]\right\}'$$

where the prime indicates differentiation with respect to η. After integration we obtain

$$S = -\tau(\eta) + D, \quad \tau = \int_0^\eta \frac{2g^2}{\eta}\,d\eta, \quad D = \text{const} \qquad (2.5)$$

$$T = \frac{1}{2}\left(\frac{\varphi}{\eta}\right)^2 + f - f(0) \qquad (2.6)$$

We substitute (2.1) — (2.5) into the first and third equations of system (1.1) and write the system of equations and boundary conditions for functions f and g in the form

$$f'' + (1+\varphi)f'/\eta - f^2 - \tau + D = 0 \qquad (2.7)$$
$$g'' + (1+\varphi)g'/\eta - (1 - \varphi + \eta^2 f)g/\eta^2 = 0 \qquad (2.8)$$
$$f' = \varphi = g = 0, \quad \eta = 0 \qquad (2.9)$$
$$f = g = 0, \quad \eta = 1 \qquad (2.10)$$
$$\varphi = \gamma = v_0 a/\nu = \text{const}, \quad \eta = 1 \qquad (2.11)$$

Parameter γ is positive for injection into and negative for suction from the pipe through its porous walls.

In the case of impermeable walls

$$\varphi = 0, \quad \eta = 1 \qquad (2.12)$$

we have the problem (2.7) — (2.12) for determining eigen functions corresponding to eigenvalues D_k.

We have obtained here two characteristic solutions of the problem formulated above. The question of the over-all number of such solutions remains open.

Note that in /1,2/ the stream function was assumed proportional to v_0 and defined by the formula

$$\psi = 2v_0 a x \varphi_*(\eta)$$

This representation of the stream function entails the impossibility of obtaining a non-trivial solution $(\psi \neq 0)$ in the case of impermeable walls when $v_0 = 0$. But the dimensional of combination νa is that of the stream function, and that of combination ν/a is that of velocity. These properties were used for writing the solution in the form (2.1) — (2.3) and, as shown below, enabled us to obtain solutions for rotational and laminar flows in a pipe with impermeable walls and damped end face.

3. Let us first consider the laminar flow $g = \tau = 0$. Solutions for small and large γ were obtained in /2/ by expanding in series in powers of parameter γ or of its inverse, respectively. Using the data of /2/, we obtain for the longitudinal velocity component on the pipe axis $f_0 = f(0)$ and constant D the expressions

$$f_0 = 4\gamma\left(1 - \frac{1}{18}\gamma + \frac{166}{10800}\gamma^2 - \ldots\right) \qquad (3.1)$$
$$D = 16\gamma\left(1 + \frac{3}{4}\gamma - \frac{11}{270}\gamma^2 + \ldots\right), \quad |\gamma| \ll 1$$

$$f_0=\gamma\left[\pi+\frac{10.602}{\pi\gamma}+O(\gamma^{-2})\right]$$
$$D=\gamma^2\left[\pi^2+\frac{20.204}{\gamma}+O(\gamma^{-2})\right],\quad |\gamma|\gg 1 \tag{3.2}$$

It was shown in /3/ that as $|\gamma|\to\infty$ there exists a multiplicity of limit solutions of Eqs.(2.7) with boundary conditions (2.9)—(2.11):

$$f=(-1)^n\pi\gamma(2n+1)\cos\left[(2n+1)\frac{\pi}{2}\eta^2\right]$$
$$D=\pi^2\gamma^2(2n+1)^2,\quad n=0,1,2\ldots \tag{3.3}$$

which for $n=0$ correspond to the first terms of expansions (3.2). The existence of solutions (3.3) indicates that the solution of this problem is not unique, at least in some domains of parameter γ variation.

Here, solutions of the problem were obtained by numerical methods.

Equation (2.7) is invariant to transformation

$$\eta_1=B\eta,\quad f_1=f/B^2,\quad \varphi_1=\varphi,\quad D_1=D/B^4,\quad B=\text{const} \tag{3.4}$$

which makes possible the reduction of the formulated boundary value problem to the Cauchy problem. It is possible to specify $f_1(0)$ and D_1, and determine γ from the calculation results.

Integration was initially carried out from $\eta_1=0$ to point $\eta_1=\eta_1^*$ at which function f_1 vanishes, i.e. satisfying condition (2.10). Integration was then extended beyond that point, since for certain values of the pair $f_1(0)$ and D_1 function f_1 vanishes for the second time already for $\eta_1=\eta_1^*>\eta_1^\circ$. For other pairs of $f_1(0)$ and D_1 the absolute value of f_1 monotonically increases to any arbitrary value. Vanishing of f_1 for the third time was not achieved.

The step h_1 of integration with respect to η_1 was constant and selected (by iterations) so as to have fifty calculation nodes, not counting point $\eta_1=0$, where the solution was obtained by expanding functions in power series within terms of order η_1^4 inclusive. The obtained in this manner f_1 and φ_1 for $\eta_1=h_1$ were used as the initial values for integration by the Kutta—Runge method in the case of $\eta_1>h_1$. The conditions for f_1 to vanish when $\eta_1=\eta_1^\circ$ or $\eta_1=\eta_1^*$ were obtained with the relative error

$$\delta=|f_1(\eta_1^\circ)|/|f_1|_{\max}<10^{-4}$$

The obtained functions f_1,φ_1 and the values of argument η_1 and of constant D were recalculated using (3.4) for quantities without subscript. The recalculation coefficient was assumed to be η_1° or η_1^* and parameter γ was determined by $\varphi_1(\eta_1^\circ)$ or $\varphi_1(\eta_1^*)$, respectively.

The dependence of f_0 and $\sigma=\sqrt{|D|}\operatorname{sign}D$ on parameter γ is shown in Fig.1, where solid lines relate to f_0 and the dash lines to σ. It will be seen that solutions of the considered here problem are not unique for all values of γ for which their derivation proved possible.

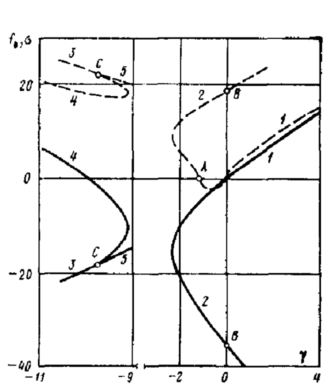

Fig.1

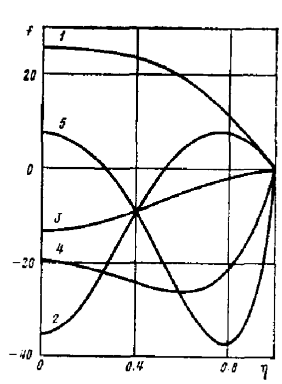

Fig.2

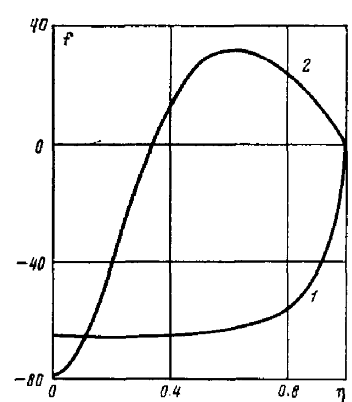

Fig.3

The values of f_0 and σ along branch 1 of solution with $0<\gamma<1$ conform with those calculated by formulas (3.1), and with increasing γ continuously approach the values obtained from expansion (3.2). When $\gamma = 7.4311$, solution 1 yields $f_0 = 25.571$, $D = 698.18$. The longitudinal velocity component profile is shown in Fig.2 by curve *1*.

When $\gamma < 0$ and $|\gamma| \ll 1$, solution 1 also conforms with (3.1) but, as $|\gamma|$ increases, it no longer approaches that of (3.2), becoming another solution (branch 2) of the problem.

Of particular interest are solutions 1 and 2 at points A and B, respectively. In the first of these $D = 0$ when $\gamma = -1.2065$ which indicates constant pressure along a porous pipe. The profile of $f(\eta)$ of solution at point A is close to that of Poiseuille. Point B corresponds to the solution of the Navier–Stokes equations for a rotational flow at the damped end of a round pipe with impermeable walls ($\gamma = 0$). At point B $f_0 = -35.315$ and $D = 348.68$. The longitudinal velocity component profile is shown in Fig.2 by curve *2*. This solution is characteristic for the boundary value problem when $\gamma = 0$.

Curve *3* in Fig.2 shows the distribution of $f(\eta)$ in the region of transition of solution 1 to solution 2 ($\gamma = -2.2469$, $f_0 = -13.385$, $D = 59.648$).

Two other branches of the solution that transform into each other were obtained with parameter $\gamma < -9.1126$. Profiles of longitudinal velocity component for the solution of branch 3 with $\gamma = -10.026$ and $f_0 = -19.765$, $D = 533.49$, and that of branch 4 with $\gamma = -11.083$ and $f_0 = 7.439$, $D = 442.70$ are shown in Fig.2 by curves *4* and *5* respectively.

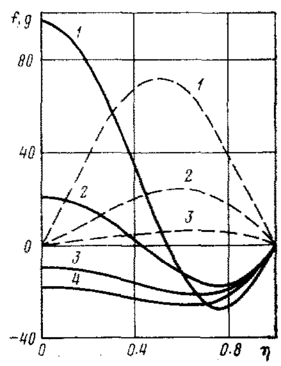

Fig.4

At point C solution 5 for the rotational flow, considered in Sect.4, branches off from solution 3 of the problem of laminar flow.

Profiles of f are also shown in Fig.3. Curve *1* relates to solution 3 with $\gamma = -27.279$ ($f_0 = -64.925$, $D = 4248.2$) and indicates division of the flow field in the inviscid main stream with $f = \text{const} \approx -\sqrt{D}$ and the boundary layer whose thickness diminishes with increasing $|\gamma|$. Curve *2* with ($\gamma = 7.3329$, $f_0 = -78.621$, $D = 1576.6$) corresponds to a flow that can be considered to be the stream generated by injection of fluid through the pipe porous walls in a direction opposite to the flow in it. Thickness of the stream also diminishes as γ increases.

In concluding the investigation of the laminar flow, we would point out that numerical calculations /4/ of the self-similar solution of the Navier–Stokes equations for flows in a plane-parallel porous channel (a is the channel half-height) have shown that this solution with $\gamma > 0$ behaves similarly to the obtained here solution 1 for a round pipe, while continuously increasing in the region of $\gamma < 0$ with increase of $|\gamma|$, it passes, unlike in the case of a round pipe, from a solution with parabolic distribution $f(\eta)$ to the solution for an inviscid flow with boundary layer, as obtained by the author in /5/.

4. Let us pass to the rotational flow. A somewhat different calculation scheme was used for obtaining the solution of this problem. The boundary value problem (2.7)–(2.12) was solved by the method of ranging. The invariance of Eqs.(2.7) and (2.8) to transformation

$$\eta_1 = B\eta, \quad f_1 = f/B^2, \quad g_1 = g/B^2, \quad \varphi_1 = \varphi, \quad D_1 = D/B^4, \quad B = \text{const} \tag{4.1}$$

enables us to reduce by one the number of ranging parameters.

Because of this, some arbitrary constant value was assigened in the process or ranging to $g_1'(0)$. The ranging parameters were $f_1(0)$ and D_1. Integration was carried out from $\eta_1 = 0$ to point $\eta_1 = \eta_1^\circ$ at which function f_1 vanished for the first or second time (depending on γ). The unknown quantities $f_1(0)$ and D_1 were determined in conformity with the stipulation that conditions $g_1(\eta_1^\circ) = 0$, $\varphi_1(\eta_1^\circ) = \gamma$ must be satisfied with required accuracy. Newton's method was used for approximating the unknown $f_1(0)$ and D_1.

The selection of step and the determination of solution in the η_1 neighborhood were as in the derivation of solution in the case of laminar flow. Conditions $f_1 = 0, g_1 = 0, \varphi_1 = \gamma$ for $\eta_1 = \eta_1^\circ$ were satisfied with the relative error $\delta = |z(\eta_1^\circ)|/|z|_{max} < 10^{-5}$, where z is any of functions f_1, g_1, φ_1.

The input functions f, g, φ and constant D were determined by the simple recalculation in conformity with (4.1) with $B = \eta_1^\circ$ as the recalculation coefficient.

The results of solution of (2.7) and (2.8) were used for calculating the quantity

$$\lambda = \int_0^1 \eta^2 f g \, d\eta$$

that defines the moment of momentum transfer

$$L = 2\pi \int_0^a r^2 u w \, dr$$

and, also, the dimensionless rotation parameter

$$\omega = \frac{a|L|}{2\pi\psi^2(a)} = \frac{|\lambda|}{\gamma^2}$$

which is a combination of the moment of momentum stream, volume flow rate through a given cross section, and of the pipe radius.

The calculated dependence of $f(0)$, $g'(0)$, D, λ, and ω on parameter γ is tabulated below

γ	$f(0)$	$g'(0)$	D	λ	ω
−9.7479	−18.209	0	484.29	0	0
−9	−14.789	6.4977	389.38	−16.851	0.20804
−7	− 3.9667	18.835	256.98	−37.337	0.76197
−5	10.685	38.022	483.33	−54.789	2.1916
−3	32.337	79.982	1662.0	−70.601	7.8446
−2	47.695	98.832	3100.9	−77.841	19.460
−1	67.941	137.18	6456.0	−84.640	84.641
0	95.511	193.35	10742	−91.086	∞
1	134.34	278.27	20433	−97.335	97.335
2	190.92	410.88	40097	−103.68	25.919

Curves of longitudinal (solid lines) and circular (dash lines) velocity components are shown in Fig.4 for $\gamma = 0, -4, -8, -9.7479$ by curves *1–4*, respectively.

Of particular interest is the solution obtained in the case of impenetrable pipe walls ($\gamma = 0$) that is, also, the eigenvalue of the boundary value problem (2.7)—(2.10), (2.12). A flow conforming to that equation can be obtained by imparting to the fluid at some reasonable distance from the damped end of the pipe a moment of momentum, for instance, by tangential injection of a stream or by a rotating impeller.

For γ close to -9.7479 the rotation parameter is small, and $f(\eta)$ retains its sign, however, due to the fairly intensive sucking, the profile of $f(\eta)$ is of saddle form. As γ approaches zero with simultaneous increase of ω, the trough in the profile of $f(\eta)$ deepens near the axis due to the change of the pressure distribution field, induced by the stream rotation. The tabulated data indicate that when $\gamma = -6.39$, to which corresponds $\omega = 1.05$, a reverse flow is generated in the region close to the axis, which increases as γ continues to approach zero. When $\gamma < 0$ the transfer of the mass of fluid as a whole is away from the device inducing rotation to the fluid, while for $\gamma > 0$ the transfer is toward the latter. However, the first case is more interesting from the practical point of view.

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